Review Session

Ch. 6: Heat Eqn.

$$\begin{cases} (\partial_t - \Delta)u = 0 & \mathbb{R}^+_t \times \mathbb{R}^n \\ u(0,x) = h(x) \end{cases}$$

- We reduced the PDE to an ODE by the time vs. space scale argument
- This gave a solution $u(t,x) = \int_{\mathbb{R}^n} H_t(x-z)h(z)dz$ for $H_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$

⇝ If $h \in C^0(\mathbb{R}^n)$, then $u(t,x) \in C^\infty((0,\infty)\times\mathbb{R}^n)$

⇝ We also had infinite propagation speed

- On the inhomogeneous problem

$$\begin{cases} (\partial_t - \Delta)u = f \\ u(0,x) = g \end{cases}$$

we saw a solution

$$u(t,x) = \int_0^t \int_{\mathbb{R}^n} H_{t-s}(x-y)f(s,y)dyds + \int_{\mathbb{R}^n} H_t(x-y)h(y)dy$$

Via Duhamel's method. (Solve $\eta_s(t,x)$ so $\eta_s(s,x) = f(s,x)$ & $(\partial_t - \Delta)\eta_s(t,x) = 0$, $u(t,x) = \int_0^t \eta_s(t,x)ds$)

- Recall Dirichlet + Neumann B.C., interpretations on a metal rod (1D Heat Eqn) & Connection to general Diffusion

Ch. 7: Function Spaces

- We defined inner products, norms, and limits in vector spaces. We used this to talk about Cauchy Sequences & Completeness, giving us Hilbert & Banach spaces.

- In Hilbert spaces, we talked about orthonormal bases: $\{e_i\}_{i=1}^{\infty}$ so $\langle e_i, e_j \rangle = \delta_j^i$ and for all $f \in \mathcal{H}$,

  $f = \sum_{i=1}^{\infty} \langle f, e_i \rangle e_i$

  For $S_n[f] = \sum_{i=1}^{n} \langle f, e_i \rangle e_i$, we said $\{e_i\}$ is a basis iff. $S_n[f] \to f$ for all $f \in \mathcal{H}$.

- For an arbitrary orthonormal set $\{c_i\}$, we established Bessel's Inequality:

  $$\sum_{j=1}^{\infty} |\langle v, c_j \rangle|^2 \leq \|v\|^2$$

  with equality iff. $S_n[v] \to v$

  ⤳ This also gave us a characterization of a basis via orthogonality: $\{e_i\}$ is a basis iff. the only $v \in \mathcal{H}$ so $\langle v, e_i \rangle = 0$ for all i is $v = 0$.

- We also established some basic measure theory with the goal of reaching L^p spaces

  $$L^p(\Omega) = \{ f : \Omega \to \mathbb{C} \text{ measurable} : \left(\int_\Omega |f|^p d\mu\right)^{1/p} = \|f\|_p < \infty \}$$

  which were Banach spaces under $\|\cdot\|_p$. In particular, $L^2(\Omega)$ was a Hilbert space.

- Recall that L^p spaces only "cared" about objects up to a set of measure 0

- We also had that for any $f \in L^p(\Omega)$, $\exists \{\psi_k\} \subset C_c^\infty(\Omega)$ such that $\psi_k \to f$ in the L^p norm.

- Lastly, we looked at cases where the Laplacian was self-adjoint in L^2 $(\langle \Delta u, v\rangle = \langle u, \Delta v\rangle)$

Ch. 8: Fourier Series

- we motivated a search for a Heat eqn. solution by a sum of product-solutions

$$u(t,x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \phi_k(x)$$

for $\phi_k(x)$ Helmholtz Solutions (on $(0,\pi)$, these looked like $\sin(kx)$)

$\rightsquigarrow$ Since these $\phi_k(x)$ looked like eigenfunctions of Δ, and could be made normal, the $\phi_k(x)$ were an orthonormal set which we wished to show was a basis

- The periodicity of these functions led us to consider $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and $L^2(\mathbb{T})$. This gave $\phi_k(x) = e^{ikx}$

we then looked at $\sum_{k \in \mathbb{Z}} a_k e^{ikx}$

for $a_k = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-ikx}\,dx$, $a_0 = 1/2\pi$

- In particular, we considered pointwise convergence

$$\sum a_k e^{ikx} = f(x) \quad \text{if} \quad \underset{y \in (x-\varepsilon, x)}{\text{ess-sup}} \left| \frac{f(x) - f(x-y)}{y} \right| < \infty$$

for some $\varepsilon > 0$.

$\rightsquigarrow$ This followed an argument writing $S_n[f](x) = (f * D_n)(x)$ for the Dirichlet kernel D_n

- Next, we looked at uniform convergence

$$\sup_{x \in \mathbb{T}} |f(x) - S_n[f](x)| \to 0 \quad \text{as } n \to \infty$$

and established that this held if $f \in C^1(\mathbb{T})$

⇝ Also recall that if $\{f_n\} \subseteq C^0(\Omega)$ & $f_n \to f$ uniformly, then $f \in C^0(\Omega)$.

- Lastly, we considered L^2 convergence, which held for all $f \in L^2$ (i.e. $\frac{1}{\sqrt{2\pi}} e^{ikx}$ gave a basis for $L^2(\mathbb{T})$).

⇝ This gave Parseval's Identity $\sum_{k \in \mathbb{Z}} |c_k[f]|^2 = \frac{1}{2\pi} \|f\|_{L^2}^2$.
for $c_k[f] = \frac{1}{2\pi} \int_{\mathbb{T}} f e^{-ikx} dx$

- We also extracted information about the regularity of $f(x)$ from the Fourier Coefficients:

$$f \in C^m(\mathbb{T}) \iff \sum_{k \in \mathbb{Z}} k^{2m} |c_k[f]|^2 < \infty$$

$$\sum_{k \in \mathbb{Z}} |k|^m |c_k[f]| < \infty \Rightarrow f \in C^m(\mathbb{T})$$

- This gave that for "nice" initial data $h(x) \in C^1(\mathbb{T})$,

$$u(t,x) = \sum c_k[h] e^{-k^2 t} e^{ikx}$$

Solved $(\partial_t - \partial_x^2) u = 0$ on $\mathbb{T}$ with

$u(0,x) = h(x)$, $\quad u \in C^\infty((0,\infty) \times \mathbb{T})$.

Ch. 9: Maximum Principles

→ We began by looking at the Laplace Eqn $\begin{cases} -\Delta u = 0 \\ u|_{\partial u} = f \end{cases}$

- On $\mathbb{D} = B(0,1) \subseteq \mathbb{R}^2$; we showed that a solution existed via

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\eta) f(\eta)\,d\eta$$

with $P_r(\theta) = \sum_{k\in\mathbb{Z}} r^{|k|} e^{ik\theta}$ the Poisson kernel

- Next, we considered a general bdd. domain $\Omega \subseteq \mathbb{R}^n$. with

$$G_R(x) = \begin{cases} \frac{1}{2\pi}\ln(r/R) & n=2 \\ \frac{1}{(n-2)A_n}\left[\frac{1}{R^{n-2}} - \frac{1}{r^{n-2}}\right] & n>2 \end{cases}$$

we established the Mean Value Formula

$$u(x_0) = \frac{1}{A_n R^{n-1}} \int_{\partial B(x_0,R)} u(x)\,dS + \int_{B(x_0;R)} G_R(x-x_0)\,\Delta u(x)\,dx$$

Then, we saw

$$-\Delta u = 0 \text{ in } \Omega \iff u(x_0) = \frac{1}{A_n R^{n-1}} \int_{\partial B(x_0,R)} u(x)\,dS$$

$$\iff u(x_0) = \frac{n}{A_n R^n} \int_{B(x_0,R)} u(x)\,dx$$

for all $x_0 \in \Omega$, $\overline{B(x_0,R)} \subseteq \Omega$

- Next, we used this to derive the strong maximum principle: $\Omega \subseteq \mathbb{R}^n$ a bdd domain and

If $-\Delta u \leq 0$ on

$u(x_0) = \max_{\Omega} u$ for $x_0 \in \Omega$, then u is constant.

This implied the weak maximum principle

$$\max_{\Omega} u = \max_{\partial\Omega} u$$

and that the Laplace eqn's solution is unique (if it exists)

• We extended this to 2 cases:

1.) If $L = -\sum_{i,j=1}^{n} a_{ij} \partial_{x_i} \partial_{x_j} + \sum_{j=1}^{n} b_j \partial_{x_j}$

for $a_{ij}, b_j \in C^0(\Omega)$, $a_{ij} = a_{ji}$, and

$\sum_{i,j=1}^{n} a_{ij}(x) v_i v_j \geq \eta \|v\|^2$ for some $\eta > 0$, all x, v, then

$Lu \leq 0$ on bdd domain $\Omega \Rightarrow \max_{\Omega} u = \max_{\partial\Omega} u$

2.) If $\frac{\partial u}{\partial t} - \Delta u \leq 0$ on $[0,\tau] \times \Omega$, then $\max_{[0,\tau]\times\bar{\Omega}} u$ occurs at (t_0, x_0) with either $t_0 = 0$ or $x_0 \in \partial\Omega$.

⇝ 2.) A.) If $\Omega = \mathbb{R}^n$ and u is bdd on any $[0,\tau] \times \mathbb{R}^n$, $\tau > 0$, then $\max_{[0,\infty)\times\mathbb{R}^n} u \leq \max_{\mathbb{R}^n} u(0,x)$

This gave some uniqueness to heat solutions.

Ch. 10: Weak Solutions

• we looked at weak derivatives $u' = f$ weakly if

$\int_\Omega f \psi \, dx = -\int_\Omega u \psi' dx$ for all $\psi \in C_c^\infty(\Omega)$

⇝ $Du \in L^1_{loc}$

⇝ if weak derivative is continuous, $u \in C^1$ & it is a strong/normal derivative.

• This supplied a way to "weakly solve"

$\begin{cases} \partial u/\partial t + \partial q/\partial x = 0 & (0,\infty) \times \mathbb{R} \\ u|_{t=0} = g \in L^1_{loc}(\mathbb{R}) \end{cases}$

by $\int_0^\infty \int_{\mathbb{R}} u \frac{\partial \psi}{\partial t} + q \frac{\partial \psi}{\partial x} \, dx dt + \int_{-\infty}^{\infty} g \psi|_{t=0} dx = 0$

for all $\psi \in C_c^\infty([0,\infty) \times \mathbb{R})$

- In the case g was piecewise continuous, we set u to be piecewise defined by g

$$u = \begin{cases} u_- & x < G(t) \\ u_+ & x > G(t) \end{cases}$$

and derived that

$$G'(t) = \frac{q(u_+) - q(u_-)}{u_+ - u_-} \quad \text{the RH-condition.}$$

$\rightsquigarrow$ Shock & Rarefaction Solutions

- Next, we defined Sobolev Spaces

$$H^m(\Omega) = \{ u \in L^2(\Omega) ;\ D^\alpha u \in L^2(\Omega) \text{ weakly for } |\alpha| \leq m \}$$

and $H_0^m(\Omega) = \overline{C_c^\infty(\Omega)}$ which encoded "trace on boundary 0"

- Via Fourier Coefficients, we noted

$H^m(\Omega) \subset C^k(\Omega)$ if $m > k + n/2$, $\Omega \subseteq \mathbb{R}^n$.

- Lastly, we gave weak formulations of our main 3 equations

$$\begin{cases} -\Delta u = \lambda u + f \\ u|_{\partial\Omega} = 0 \\ \text{weakly} \end{cases} \iff u \in H_0^1(\Omega) \quad \int_\Omega \nabla u \cdot \nabla \psi - \lambda u \psi - f \psi \, dx = 0 \text{ for } \psi \in C_c^\infty(\Omega)$$

$$\begin{cases} (\partial_t^2 - \Delta) u = 0 \\ u|_{\partial\Omega} = 0 \\ u|_{t=0} = g \\ \partial_t u|_{t=0} = h \\ \text{weakly} \end{cases} \iff \begin{aligned} & u(t,\cdot) \in H_0^1(\Omega),\ \|u(t,\cdot)\|_{H^1} \text{ integrable in } t \\ & \int_0^\infty \int_\Omega u \partial_t^2 \psi + \nabla u \cdot \nabla \psi \, dx\, dt = \\ & - \int_\Omega g \, \partial_t \psi|_{t=0} dx + \int_\Omega h \psi|_{t=0} dx \end{aligned}$$

$$\begin{cases} (\partial_t - \Delta) u = 0 \\ u|_{\partial\Omega} = 0 \\ u|_{t=0} = h \\ \text{weakly} \end{cases} \iff \begin{aligned} & u(t,\cdot) \in H_0^1(\Omega),\ \|u(t,\cdot)\|_{H^1} \text{ int. in } t \\ & \int_0^\infty \int_\Omega -u \frac{\partial \psi}{\partial t} + \nabla u \cdot \nabla \psi \, dx\, dt = \int_\Omega h \psi|_{t=0} dx \end{aligned}$$

~~$\rightsquigarrow$ Note, $u \in H_0^1([0,\infty) \times$~~ $\rightsquigarrow$ Note, $u \in H_0^1([0,\infty) \times \Omega)$ is also sufficient? However, it encodes different data.